A GENERALIZATION OF BERNOULLI’S DIFFERENTIAL EQUATION

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ABSTRACT

It is well known that the Bernoulli’s differential equation has the form \( Y' + p(x)Y = q(x)Y^\alpha \) where \( \alpha \) is a fixed real number. In this paper, under certain conditions, we give a generalization of this equation when we change \( \alpha \) for an adequate function \( r(x) \).

1. INTRODUCTION

It is well known the Bernoulli’s differential equation and how to solve it, which is given by [1-3]:

\[
Y' + p(x)Y = q(x)Y^\alpha, \tag{1}
\]

where \( \alpha \) is a fixed real number. In this work, we shall study the differential equation:

\[
Y' + p(x)Y = q(x)Y^{r(x)}, \tag{2}
\]

for certain functions \( p(x), q(x) \) and \( r(x) \). For this, first we will see some types of differential equations with the structure:

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as we can see in Theorem 1. A solution \( y = y(x) \) of the previous differential equation shall be represented by a series of powers, as in (4).

In reality, we will assume that all functions can be expanded in series of powers on an interval \( I = (-t, t) \), with \( t > 0 \), and all the products between them will be convergent. Therefore, we will not bother to mention this.

2. Solution of a differential equation expressed in series of powers

Let \( \{f_n\}_{n \geq 0} \) and \( y \) functions. We write for each \( n \geq 0 \):

\[
f_n(x) = \sum_{l=0}^{\infty} a_{n,l} x^l,
\]

and

\[
y = \sum_{m=0}^{\infty} b_m x^m
\]

Then, using the Cauchy product, we obtain:

\[
y^n = \sum_{m=0}^{\infty} c_{n,m} x^m
\]

where

\[
c_{n,m} = \sum_{s_1 + \cdots + s_n = m} b_{s_1} \cdots b_{s_n}
\]

for all \( n \geq 1 \) and \( m \geq 0 \). We define

\[
c_{0,0} := 1 \quad \text{and} \quad c_{0,m} := 0
\]

for all \( m \geq 1 \). Thus, we have:

**Theorem 1.** Under the above conditions and notations, we have that \( y \) is solution of the differential equation:
\[ Y' = \sum_{n=0}^{\infty} f_n(x) Y^n \]  

(8)

if and only if the coefficients of the representation of \( y \) as series of powers in (4) satisfy that \( b_0 = y(0) \) and for each \( m \geq 0 \):

\[ b_{m+1} = \frac{1}{m+1} \sum_{k=0}^{m} \left( \sum_{n=0}^{\infty} a_{n,k} c_{n,m-k} \right). \]  

(9)

**Proof.** We have that \( y \) is solution of (8) if and only if it holds:

\[
\sum_{m=0}^{\infty} (m + 1)b_{m+1} x^m = y' = \sum_{n=0}^{\infty} f_n(x) y^n \\
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{\infty} a_{n,l} x^l \right) \left( \sum_{m=0}^{\infty} c_{n,m} x^m \right) \\
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_{n,k} c_{n,m-k} \right) x^m \\
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} \sum_{k=0}^{m} a_{n,k} c_{n,m-k} \right) x^m \\
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} \sum_{k=0}^{m} a_{n,k} c_{n,m-k} \right) x^m \\
= \sum_{m=0}^{\infty} (m + 1)b_{m+1} x^m = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} a_{n,k} c_{n,m-k} \right) x^m.
\]

Hence, \( y \) is solution of the differential equation (8) if and only if:

\[ b_{m+1} = \frac{1}{m+1} \sum_{k=0}^{m} \left( \sum_{n=0}^{\infty} a_{n,k} c_{n,m-k} \right). \]
for all \( m \geq 0 \), where each \( c_{n,m-k} \) is given as in (6). This completes the proof.

In particular, from (9), we have that \( b_0 = y(0) \),

\[
b_1 = \sum_{n=0}^{\infty} a_{n,0} b_0^n, \tag{10}\]

and

\[
b_2 = \frac{1}{2} \left( b_1 \sum_{n=1}^{\infty} a_{n,0} b_0^{n-1} + \sum_{n=0}^{\infty} a_{n,1} b_0^n \right). \tag{11}\]

**Corollary 2.** Let \( f_0, f_1 \) and \( g \) functions, and:

\[
\sum_{n=2}^{\infty} a_n \tag{12}\]

a series absolutely convergent. We suppose that \( f_0 \) and \( f_1 \) are given as in (3), and we write:

\[
g(x) = \sum_{i=0}^{\infty} u_i \cdot x^i. \tag{13}\]

Then, \( y \) is solution of the differential equation:

\[
Y' = f_0(x) + f_1(x)Y + g(x) \sum_{n=2}^{\infty} a_n Y^n, \tag{14}\]

if and only if the coefficients of the representation of \( y \) as series of powers in (4) are given by \( b_0 = y(0) \) and for each \( m \geq 0 \):

\[
b_{m+1} = \frac{1}{m+1} \sum_{k=0}^{m} \left( a_{0,k} c_{0,m-k} + a_{1,k} c_{1,m-k} + \sum_{n=2}^{\infty} a_n u_k c_{n,m-k} \right). \tag{15}\]

**Proof.** We define for each \( n \geq 2 \), \( f_n(x) := a_n g(x) \) for all \( x \in (-t, t) \), that is \( f_n \) is given as (3) such that:

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\[ a_{n,l} = a_n u_l, \]  

(16)

for each \( l \geq 0 \) \((n \geq 2)\). Then, applying the Theorem 1, we obtain (15) from (9).

\[ \text{Example 3. In the conditions and notations of Corollary 2, we have that } y \text{ is solution of the differential equation:} \]

\[ Y' = f_0(x) + f_1(x)Y + g(x) \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} Y^n, \]  

(17)

if and only if the coefficients of the representation of \( y \) in (4) are given by \( b_0 = y(0) \) and for each \( m \geq 0 \):

\[ b_{m+1} = \frac{1}{m+1} \sum_{k=0}^{m} \left( a_{0,k} c_{0,m-k} + a_{1,k} c_{1,m-k} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} u_k c_{n,m-k} \right). \]  

(18)

\[ \text{3. A particular differential equation} \]

We conserve the conditions and notations of the Section 2. Also, we write:

\[ h(x) = \sum_{l=0}^{\infty} \nu_l x^l, \]  

(19)

thus, we have:

\[ \text{Theorem 4. We assume } |y(x) - 1| < 1 \text{ for all } x \in (-t, t). \text{ Then } y \text{ is solution of the differential equation:} \]

\[ Y' = f_0(x) + h(x)Y + g(x)Y \ln(Y), \]  

(20)

if and only if the coefficients of \( y \) in its representation of series of powers in (4) are given by \( b_0 = y(0) \) and, for each \( m \geq 0 \), \( b_{m+1} \) is given by the equation (15) of Corollary 2, where for each \( n \geq 2 \):
\[ a_n = \sum_{m=1}^{\infty} a_{n-1,m} \quad (21) \]

with

\[ a_{n,m} := \begin{cases} \frac{(-1)^{n+1}}{m} \binom{m}{n} & \text{if } 0 \leq n \leq m \\ 0 & \text{if } n > m \end{cases} \quad (22) \]

**Proof.** We have that:

\[ \ln(y) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (y - 1)^m, \quad (23) \]

because \(|y(x) - 1| < 1\) for all \(x \in (-t, t)\). Hence:

\[ y \ln(y) = \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} a_{n-1,m} \right) y^n, \quad (24) \]

such that:

\[ a_{n,m} := \begin{cases} \frac{(-1)^{n+1}}{m} \binom{m}{n} & \text{if } 0 \leq n \leq m \\ 0 & \text{if } n > m \end{cases} \]

Then, the differential equation in (20) is equivalent to the following equation:

\[ Y' = f_0(x) + f_1(x)Y + g(x) \sum_{n=2}^{\infty} \left( \sum_{m=1}^{\infty} a_{n-1,m} \right) Y^n \quad (25) \]

where:

\[ f_1(x) = h(x) + \left( \sum_{m=1}^{\infty} a_{n-1,m} \right) g(x), \quad (26) \]

therefore, the affirmation is followed from Corollary 2.
4. Bernoulli’s generalized differential equation

We consider Bernoulli’s generalized differential equation (2), that is:

\[ Y' + p(x)Y = q(x)Y^{r(x)}, \]

where \( r(x) \neq 1 \) for all \( x \in I \). We note that when \( r(x) \) is constant on \( I \), then the differential equation is the standard differential equation of Bernoulli.

Thus, under the substitution:

\[ u = y^{1-r(x)}, \tag{27} \]

we have the relationship:

\[ \ln(u) = (1 - r(x)) \ln(y). \tag{28} \]

Using the equation (28) and Bernoulli’s generalized differential equation (2), we obtain:

\[ \frac{u'}{u} = \frac{d \ln(u)}{dx} = (1 - r(x))q(x) \frac{1}{u} - p(x)(1 - r(x)) - \frac{r'(x)}{1 - r(x)} \ln(u), \tag{29} \]

or equivalently:

\[ u' = (1 - r(x))q(x) - p(x)(1 - r(x))u - \frac{r'(x)}{1 - r(x)} u \ln(u). \tag{30} \]

We note that the equation (30) establishes that \( u \) is solution of the differential equation given in Theorem 4, equation (20), where \( f_{0}(x) = (1 - r(x))q(x) \), \( h(x) = -p(x)(1 - r(x)) \) and \( g(x) = -r'(x)/(1 - r(x)) \). Therefore, we know how is given the function \( u \) (Theorem 4). But, as \( u = y^{1-r(x)} \) (equation (27)), then we have:

**Theorem 5.** We assume that \( |r(x)| < 1 \), for all \( x \in (-r, r) \). The function \( y \) given in (4) is solution of the Bernoulli’s generalized differential equation if and only if:

\[ y = \sum_{n=0}^{\infty} \left( \frac{1}{n!} \left( \sum_{i=0}^{\infty} r(x)^i \right)^n \left( \sum_{k=0}^{\infty} \beta_k u^k \right)^n \right), \tag{31} \]

where for all \( k \geq 0 \):

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\[ \beta_k = \sum_{m=1}^{\infty} \alpha_{k,m}, \quad (32) \]

and each \( \alpha_{k,m} \) is given by the equation (22).

**Proof.** By the equation (27), we have:

\[
y = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \ln(u) \right)^n (1 - r(x))
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right) \left( \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left( u - 1 \right)^m \right)^n
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right) \left( \sum_{m=1}^{\infty} \left( \sum_{k=0}^{m} \frac{(-1)^{k+1}}{m} \binom{m}{k} u^k \right)^n \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right) \left( \sum_{m=1}^{\infty} \left( \sum_{k=0}^{m} \alpha_{k,m} u^k \right)^n \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right) \left( \sum_{k=0}^{\infty} \left( \sum_{m=1}^{\infty} \alpha_{k,m} u^k \right)^n \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right) \left( \sum_{k=0}^{\infty} \beta_k u^k \right)^n
\]

that is:

\[
y = \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right) \left( \sum_{l=0}^{\infty} r(x)^l \right)^n \left( \sum_{k=0}^{\infty} \beta_k u^k \right)^n,
\]

this completes the proof.
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